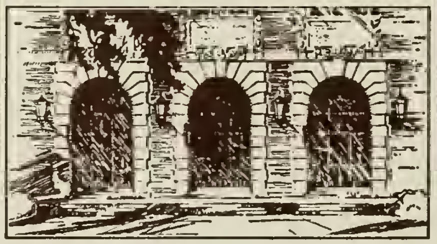

REMARKS ON THE ROUND-OFF ERRORS IN ITERATIVE PROCESSES FOR FIXED-POINT COMPUTERS *

by

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Abstract

This report develops some ideas suggested by A. H. Taub on the round-off errors in iterative processes. It will be shown that in certain cases the convergence of the process can be improved by using special method for rounding. Part I is concerned with simple first order processes. In part II, the Aitken's δ^2 process will be considered.

PART I. ERRORS IN FIRST ORDER ITERATIVE PROCESSES

Introduction

Let $G^{(1)}, \ldots G^{(m)}$ be m real functions of the real variables $x^{(1)}, \ldots x^{(m)}$. For any set of m numbers $p^{(1)}, \ldots p^{(m)}$, we shall use the vectorial notations:

$$\vec{p} = (p^{(1)}, \ldots p^{(m)}) ;$$

$$|\vec{p}| = \sqrt{(p^{(1)})^2 + \ldots (p^{(m)})^2} .$$

We consider the iterative process

$$\vec{x}_{n+1} = \vec{G}(\vec{x}_n) , \quad n = 0, 1, \ldots \tag{1}$$

and suppose there exists a vector $\vec{r}$ and a number b $(0 \leq b < 1)$ such that

$$|G(\vec{x}) - \vec{r}| \leq b \; |\vec{x} - \vec{r}| ; \tag{2}$$

the condition (2) insures the convergence of the $\vec{x}_n$'s to $\vec{r}$.

We want to realize the process (1) as a fixed-point computer under the two conditions: a) for representing each of the $x_n^{(i)}$, we use only one "word"; we consider the content of the word as an integer; b) we may use higher precision for computing the values of the functions $G^{(1)}, \ldots G^{(m)}$.

We distinguish two types of errors:

1) truncation errors; even using double precision, we cannot expect to evaluate the functions $G^{(i)}$ exactly;

2) round-off errors; according to the condition a), the value found for $G^{(i)}$ must be rounded to an integer.

Truncation errors

Let $H^{(1)}(\vec{x}), \ldots H^{(m)}(\vec{x})$ approximate the functions $G^{(1)}(\vec{x}), \ldots G^{(m)}(\vec{x})$:

$$H^{(i)}(\vec{x}) = G^{(i)}(\vec{x}) + \xi^{(i)}(\vec{x});$$

$\xi^{(i)}(\vec{x})$ is called the truncation error; it is supposed to satisfy the inequality

$$|\xi^{(i)}(\vec{x})| \leq a^{(i)}; \quad a^{(i)} = \text{constant}. \tag{3}$$

The iterative process

$$\vec{V}_{n+1} = \vec{H}(\vec{V}_n) \tag{4}$$

is considered as an approximation of (1) and gives some information about $\vec{r}$.

Theorem 1. For any $\vec{V}_1$, the sequence $\vec{V}_n$ given by (4) is bounded and all its points of accumulation $\vec{V}$ satisfy the inequality

$$|\vec{V} - \vec{r}| \leq \frac{|\vec{a}|}{1-b} \qquad \vec{a} = (a^{(1)}, \ldots a^{(m)})$$

Theorem 2. The process (4) is the best possible in the following sense: for given $\vec{a}$ and b, there exist m functions $H^{(1)}(\vec{x}), \ldots H^{(m)}(\vec{x})$ for which it is impossible to find an algorithm using only $\vec{H}$, $\vec{a}$, b, providing closer points of accumulation to r than the algorithm (4)

Proof: Let $\vec{G}(\vec{x}) = b\vec{x} + \vec{a}$,

$$\vec{H}(\vec{x}) = b\vec{x}$$

$$\vec{G}'(\vec{x}) = b\vec{x} - \vec{a}.$$

$\vec{H}(\vec{x})$ is an approximation for both $\vec{G}(\vec{x})$ and $\vec{G}'(\vec{x})$ with limits $\vec{r} = \frac{\vec{a}}{1-b}$ and $\vec{r}' = \frac{-\vec{a}}{1-b}$

If any sequence $\vec{W}_n$ has a point of accumulation $\vec{W}$ such that

$$|\vec{W} - \vec{r}| < \frac{|\vec{a}|}{1-b} ,$$

then by the triangular inequality,

$$|\vec{W} - \vec{r}'| > \frac{|\vec{a}|}{1-b}$$

and the process (4) presides in this case a better information.

Round-off errors

For the computer, the process (1) can be written in the form

$$y_{n+1}^{(i)} = [G^{(i)}(\vec{y}_n) + \xi_n^{(i)}]_R ; \tag{5}$$

$y_n^{(i)}$ is an integer.

$[\ ]_R$ is called a rounding procedure. $[x]_R$ is any integer function of x satisfying the inequality:

$$|[x]_R - x| < 1.$$

We consider two particular types of rounding procedures:

1) normal rounding: $[x]_N = [x + 0.5]$;

2) anomalous rounding: $[x]_A$: for $|x| \leq 1$, $|[x]_A| \geq |x|$

for $|x| \geq 1$, $|[x]_A| \leq |x|$

Theorem 3. Let $\vec{G}$ and $\vec{\xi}$ satisfy the equations (2) and (3). If

$$y_{n+1}^{(i)} = [G^{(i)}(y_n) + \xi_n^i]_N , \quad i = 1, 2 \dots m, \tag{6}$$

then for any y_o, there exists N such that

$$|\vec{y}_n - \vec{r}| \leq \frac{|\vec{a}|}{1-b} + \frac{\sqrt{m}}{2(1-b)} \qquad \text{for } n > N ;$$

furthermore, for given $\vec{a}$ and b, there exist a function G and the errors $\vec{\xi}$ for which the bound is attained.

Now, we restrict ourselves to the particular case $m = 1$, i.e., the process (1) becomes scalar. Equations (1), (2), (3), and (5) can be written as:

$$x_{n+1} = G(x_n) \tag{7}$$

$$|G(x) - r| \leq b\ |x-r| \tag{8}$$

$$y_{n+1} = [G(y_n) + \xi_n]_R \tag{9}$$

$$|\xi_n| \leq a \tag{10}$$

Theorem 4. Let $G(x)$ and ξ satisfy the equations (8) and (10). If

$$y_{n+1} = y_n + [G(y_n) + \xi_n - y_n]_A\ , \tag{11}$$

then for any y_o, there exists N such that

$$|y_{n+1} - r| < \frac{a}{1-b} + 1 \qquad \text{for } n > N.$$

Let us compare the theorem 4 with the theorem 3 for $m = 1$. In both cases, the bounds of errors have a common part which can be recognized from theorems 1 and 2 as provided by the truncation errors. The part due to the round-off errors is independent of b for the anomalous rounding; in particular, if $a = 0$, the error is less than 1 and if the limit r is an integer, it is reached after a finite number of steps. When the convergence is slow, i.e., $b \sim 1$, the errors can be very large for the normal rounding, even if $a = 0$; however, if $b < 0.5$, the normal rounding provides slightly better results than the anomalous rounding.

Remark. The condition (2) insures a first-order convergence for the process (1). If we assume higher convergence, i.e., if

$$|\vec{G}\ (\vec{x}) - \vec{r}| \leq b\ |\vec{x} - \vec{r}|^p\ , \qquad p > 1,$$

we get results which are quite similar, but generaly not simple to formulate. Rather roughly, the theorem 4 becomes: if y_n is computed by (11), then

$$|y_n - r| < B + 1,$$

where B is due to the truncation error.

Proofs

Lemma. Let $\vec{V}_o$ and $\vec{V}_1$ satisfy the equation (4) under the assumption (2) and (3).

a) if $|\vec{V}_o - \vec{r}| \leq \frac{|\vec{a}|}{1-b}$, then $|\vec{V}_1 - \vec{r}| \leq \frac{|\vec{a}|}{1-b}$

b) if $|\vec{V}_o - \vec{r}| > \frac{|\vec{a}|}{1-b}$, then $|\vec{V}_1 - \vec{r}| < |\vec{V}_o - \vec{r}|$

Proof. Since $\vec{V}_1 = \vec{G}\,(\vec{V}_o) + \vec{\xi}_o$:

$$|\vec{V}_1 - \vec{r}| \leq |\vec{G}(\vec{V}_o) - \vec{r}| + |\vec{\xi}_o| \leq b\ |\vec{V}_o - \vec{r}| + |\vec{a}| \tag{12}$$

a) $|\vec{V}_o - \vec{r}| \leq \frac{|\vec{a}|}{1-b}$; we have by (12):

$|\vec{V}_1 - \vec{r}| \leq |\vec{a}| \left\{ \frac{b}{1-b} + 1 \right\} = \frac{|\vec{a}|}{1-b}$, q.e.d.

b) $|\vec{V}_o - \vec{r}| > \frac{|\vec{a}|}{1-b}$; we have by (12):

$|\vec{V}_1 - \vec{r}| \leq |\vec{V}_o - \vec{r}| - (1-b)\ |\vec{V}_o - \vec{r}| + |\vec{a}| < |\vec{V}_o - \vec{r}| - |\vec{a}| + |\vec{a}| = |\vec{V}_o - \vec{r}|$;

q.e.d.

Proof of the Theorem 1. 1st Case: there is N such that $|\vec{V}_N - \vec{r}| \leq \frac{|\vec{a}|}{1-b}$;

by lemma a the same inequality holds for all $n > N$ and the theorem is proved.

2nd Case: for all $n = 0, 1, 2, \ldots$: $|\vec{V}_n - \vec{r}| > \frac{|\vec{a}|}{1-b}$;

by lemma b, the positive sequence $|\vec{V}_n - \vec{r}|$ is monotone decreasing and converges therefore to a limit ℓ.

Suppose that $\ell = \frac{|\vec{a}|}{1-b} + d$ where $d > 0$; since $b < 1$, there exists $\vec{y}_n$ such that $|\vec{y}_n - \vec{r}| < \frac{|\vec{a}|}{1-b} + \frac{d}{b}$; by 12:

$|\vec{V}_1 - \vec{r}| < \frac{b}{1-b}\ |\vec{a}| + d + |\vec{a}| = \frac{|\vec{a}|}{1-b} + c = \ell$, which is a contradiction.

Proof of the Theorem 3. Since $|[x]_N - x| \leq 0.5$, we can write the equation (6) in the form

$$y_{n+1}^{(i)} = G^{(i)}(\vec{y}_n) + \eta_n^{(i)} \qquad i = 1, 2, \ldots m$$

where $$|\eta_n^{(i)}| < a^{(i)} + 0.5 ,$$

and therefore $|\vec{\eta}_n| \leq |\vec{a}| + 0.5\sqrt{m}$.

Replacing $\vec{\xi}_n$ by $\vec{\eta}_n$ and $|\vec{a}|$ by $|\vec{a}| + 0.5\sqrt{m}$, we can apply the theorem 1: for any ϵ, there exists N such that

$$|\vec{y}_n - \vec{r}| < \frac{|\vec{a}| + 0.5\sqrt{m}}{1 - b} \qquad \text{for } n > N;$$

but since the $y_n^{(i)}$'s are integers, there exists a particular ϵ for which the preceding inequality implies

$$|\vec{y}_n - \vec{r}| \leq \frac{|\vec{a}| + 0.5\sqrt{m}}{1 - b} \text{ for } n > N, \text{ as desired.}$$

We have still to show an example valid for every $\vec{a}$ and b where the bound of error is attained. Let

$$G^{(i)}(\vec{x}) = bx^{(i)} - a^{(i)} - 0.5$$

and suppose that for the particular vector $\vec{y}_0 = 0$ we have $\vec{\xi}_0 = \vec{a}$.

Then $$\vec{y}_n = 0 \text{ and } |\vec{y}_n - \vec{r}| = \frac{|\vec{a}| + \sqrt{m} \cdot 0.5}{1 - b} \qquad \text{for } n \geq 0.$$

<u>Proof of the Theorem 4</u>. We use the two simple properties of the anomalous rounding procedures:

1) $x - 1 < [x]_A < x + 1$

2) if $p < x < q$ and $q - p > 1$, then

$$p < p + [x - p]_A < q \quad \text{, provided that p is an integer,}$$
$$\text{and } p < q + [x - q]_A < q \quad \text{, provided that q is an integer.}$$

Since the y_n's are integers, the theorem results from the three statements:

I. if $|y_0 - r| \leq \frac{a}{1-b}$, then $|y_1 - r| < \frac{a}{1-b} + 1$;

II. if $\frac{a}{1-b} < |y_0 - r| < \frac{a}{1-b} + 1$, then $|y_1 - r| < \frac{a}{1-b} + 1$;

III. if $|y_0 - r| \geq \frac{a}{1-b} + 1$, then $|y_1 - r| < |y_0 - r|$

Statement I: by lemma a:

$$r - \frac{a}{1-b} \leq y_o + G(y_o) + \xi_o - y_o \leq r + \frac{a}{1-b} ;$$

by property 1: $r - \frac{a}{1-b} - 1 < y_o + [G(y_o) + \xi_o - y_o]_A < r + \frac{a}{1-b} + 1,$

i.e. $|y_1 - r| < r + \frac{a}{1-b} + 1$, q.e.d.

Statement II: We suppose $r + \frac{a}{1-b} < y_o < r + \frac{a}{1-b} + 1$ (the proof is analogous when $r - \frac{a}{1-b} - 1 < y_o < r - \frac{a}{1-b}$) ; by lemma b:

$$p \equiv r - \frac{a}{1-b} - 1 < y_o + G(y_o) + \xi_o - y_o < y_o \equiv q ;$$

Since $y_o > r$, $q - p > 1$ and we apply the property 2:

$$r - \frac{a}{1-b} - 1 < y_o + [G(y_o + \xi_o - y_o]_A < y_o < r + \frac{a}{1-b} + 1$$

i.e., $|y_1 - r| < r + \frac{a}{1-b} + 1$, q.e.d.

Statement III: We suppose $y_o \geq r + \frac{a}{1-b}$ (the proof is analogous when $y_o \leq r - \frac{a}{1-b}$) ; by lemma b:

$$p \equiv 2r - y_o < y_o + G(y_o) + \xi_o - y_o < y_o \equiv q ;$$

by property 2, since $q - p > 1$:

$$2r - y_o < y_o + [G(y_o) + \xi_o - y_o]_A < y_o$$

i.e., $|y_1 - r| < |y_o - r|$, q.e.d.

PART II. ROUND-OFF ERRORS IN THE AITKEN'S δ^2 PROCESS

Let G(x) a real continuous function of the real variable x such that the sequence x_n defined by

$$x_{n+1} = G(x_n) \tag{1}$$

converges to the limit x = r.

By the Aitken's δ^2 process, we define another sequence:

$$\left.\begin{aligned} V_{3n+1} &= G(V_{3n}) \\ V_{3n+2} &= G(V_{3n+1}) \\ V_{3n+2} &= \frac{V_{3n}V_{3n+2}-V_{3n+1}^2}{V_{3n}+V_{3n+2}-2V_{3n+1}} \end{aligned}\right\} \tag{2}$$

Let us suppose we want to realize the process 2 on a <u>fixed-point computer</u> with the following conditions: a) We use only one "word" for representing the V_i's; we may consider the content of the word as an <u>integer</u>; b) we may use higher precision for computing $G(V_i)$.

We cannot expect to compute $G(V_i)$ without error; furthermore, if using higher precision, the result must be rounded to an integer.

<u>Definition</u>. A <u>rounding procedure</u> denoted by $[x]_R$ is any function of the real variable x satisfying the inequality

$$|[x]_R - x| < 1.$$

We shall use the following particular rounding procedures:

1) $[x]^{\nearrow}$: <u>rounding away from zero</u>; it is defined by the inequality $|[x]^{\nearrow}| \geq |x|$

2) $[x]^{\swarrow}$: <u>rounding toward zero</u>; it is defined by the inequality $|[x]^{\swarrow}| \leq |x|$

<u>Example</u>. Let G(x) = 7/8 x and V_o = 8; by (2), we have

$V_1 = 7$

$V_2 = 6{,}125$

$V_3 = 0.$

If we want to represent the V_i's only by integers and if we use the normal rounding procedure, we shall find:

$$\bar{V}_1 = 7$$

$$\bar{V}_2 = 6$$

$$\bar{V}_3 = \infty$$

It will be shown that this situation can be improved by using the following integer process:

$$\left.\begin{aligned}
W_{3n+1} &= W_{3n} + [G(W_{3n}) + \xi_{3n} - W_{3n}]^{\nearrow} \\
W_{3n+2} &= W_{3n} + [G(W_{3n+1}) + \xi_{3n+1} - W_{3n}]^{\swarrow} \\
W_{3n+3} &= W_{3n} + \left[\frac{(W_{3n} - W_{3n+1})^2}{2W_{3n+1} - W_{3n} - W_{3n+2}}\right]^{\swarrow}
\end{aligned}\right\} \qquad (3)$$

ξ_{3n} and ξ_{3n+1} are the errors of computation of $G(W_{3n})$ and $G(W_{3n+1})$; since the numerator and the denominator are integers, it is possible with the help of the remainder to compute W_{3n+3} without any error; if the numerator and the denominator are simultaneously equal to zero, then $W_{3n} = W_{3n+1} = W_{3n+2}$ and we set $W_{3n+3} = W_{3n}$.

<u>Theorem 1</u>. We suppose there exists the numbers $0 \leq b < 1$, $0 \leq c < 1$, $\delta \geq 0$, $\ell > 1$, such that:

1) $|x_1 - r| \leq b\,|x_o - r|$

   where x_o and x_1 satisfy the relation (1) and $r - \ell \leq x_o \leq r + \ell$.

2) $|V_3 - r| \leq c\,|V_o - r|$

   where V_o and V_3 satisfy the relations (2) and $r - \ell \leq V_o \leq r + \ell$.

3) $|G(x) - G(y)| \leq \delta\,|x - y|$

   where $r - \ell \leq x, y \leq r + \ell$.

4) The errors ξ_{3n} and ξ_{3n+1} in (3) satisfy the inequality

$$|\xi_j| \leq a \leq d = \frac{1}{4}\,\frac{(1-b)^2(1-c)}{(1+c)(1+\delta)}\,.$$

Then, for any W_o belonging to the interval $[r - \ell + 1, r + \ell - 1]$, there exists a finite number N such that

$$|W_{3n} - r| < 1 + \frac{a}{1-b} \quad \text{for } n > N.$$

Theorem 2. We make the assumptions:

1) The convergence of the process (1) is alternating, i.e. for $r - \ell \leq x \leq r + \ell$:

$$0 \leq r - G(x) < x - r \qquad \text{if} \qquad x - r > 0,$$

$$0 \leq G(x) - r < r - x \qquad \text{if} \qquad x - r < 0,$$

$$G(x) = r \qquad \text{if} \qquad x = r;$$

2) The errors ξ_{3n} and ξ_{3n+1} in (3) satisfy the inequality

$|\xi_j| \leq a \leq \frac{1}{3}$, where a is a fixed number.

Then, for any W_o belonging to the interval $[r - \ell + \frac{4}{3}, r + \ell - \frac{4}{3}]$, there exists a finite number N such that

$$|W_{3n} - r| \leq 1 + a \text{ for } n > N.$$

Remark. The assumption (1) of the theorem 2 is sufficient for providing the convergence of the V_n's satisfying the equations (2) for any $r - \ell \leq V_o \leq r + \ell$.

It is easy to prove the inequality

$$|V_{3n} - r| < \frac{|V_o - r|}{3^n}$$

Proof of the Theorem 1

Notations

r is the root of the equation $F(x) = 0$ and the limit of the process (1).

W_o, W_1, W_2, W_3 satisfy the equations (3) with errors ξ_o and ξ_1.

u_1, u_2, u_3 are defined by

$$\left.\begin{aligned} u_1 &= G(W_o) \\ u_2 &= G(W_1) \\ u_3 &= \frac{W_oW_2 - W_1^2}{W_o+W_2-2W_1} \; . \end{aligned}\right\}$$

The integers p, q, s, t are defined by

$q = p + 1; \quad s = q + 1; \quad t = s + 1; \quad q < r \leq s.$

Lemas

The following lemmas, except lemma 1 and 2, are valid only under the assumptions of the theorem 1.

Lemma 1. The relations 3 are invariant for the transformation

$$W_i' = -W_i \;, \quad G'(x) = -G(x) \;, \quad \xi_i' = -\xi_i \;,$$

i.e., if W_1', W_2', W_3' are computed from $W_o' = -W_o$ by replacing G by G' and ξ_i by ξ_i' in 3, then $W_1' = -W_1$, $W_2' = -W_2$, $W_3' = -W_3$.

Proof.
$$\begin{aligned} W_1' &= W_o' + [G'(W_o') - W_o' + \xi_o']^{\nearrow} \\ &= W_o' + [-(G(W_o) - W_o + \xi_o)]^{\nearrow} \\ &= -W_o - [G(W_o) - W_o + \xi_o]^{\nearrow} = - W_1; \end{aligned}$$

the proofs, based on the properties

$$[-x]^{\nearrow} = -[x]^{\nearrow} \quad \text{and} \quad [-x]^{\swarrow} = -[x]^{\swarrow}$$

and the same for W_2' and W_3' .

Lemma 2. Let x_o, x_1, x_2 and real numbers and $x_3 = \dfrac{x_ox_2 - x_1^2}{x_o + x_2 - 2x_1}$

a) $$\frac{\partial x_3}{\partial x_o} \geq 0 \; ; \quad \frac{\partial x_3}{\partial x_2} \geq 0; \quad \frac{\partial x_3}{\partial x_1} = 2\,\frac{(x_1 - x_o)(x_1 - x_2)}{(x_o + x_2 - 2x_1)^2} \; .$$

b) If $x_o > x_1$, there is the following scheme of variations for x_3, as function of x_2:

x_2	$-\infty$	$\nearrow$	$2x_1 - x_o$	$\nearrow$	x_1	$\nearrow$	x_o	$\nearrow$	$+\infty$
x_3	x_o	$\nearrow$	$\pm\infty$	$\nearrow$	x_1	$\nearrow$	$\frac{1}{2}(x_o + x_1)$	$\nearrow$	x_o

c) If $x_o > x_2 > x_1$, then $x_2 > x_3 > x_1$;

if $x_o = x_2$, then $x_3 = \frac{1}{2}(x_o + x_1)$;

if $x_1 = x_2$, then $x_3 = x_1$.

d) Considering x_2 as a function of x_o, x_1, x_2, one has:

$$x_2 = \frac{x_1^2 + x_o x_3 - 2x_1 x_3}{x_o - x_3} \; ; \quad \frac{\partial x_2}{\partial x_o} \leq 0 \; ; \quad \frac{\partial x_2}{\partial x_3} \geq 0 \; .$$

<u>Lemma 3</u>. If $W_o \geq r + 1$, then $W_1 < W_o$.

<u>Proof</u>. By assumption 1:

$$u_1 - r \leq b(W_o - r)$$

$$u_1 \leq W_o - (1 - b)(W_o - r) \leq W_o - (1 - b)$$

$$u_1 + \xi_1 \leq W_o - (1 - b) + a \leq W_o - (1 - b)\left(1 - \frac{1}{4}\,\frac{(1-b)(1-c)}{(1+c)(1+\delta)}\right) < W_o$$

$$W_1 = W_o + [u_1 + \xi_1 - W_o]^{\nearrow} < W_o \quad \text{q.e.d.}$$

<u>Lemma 4</u>. Let $V_o \geq r + 1$, V_1, V_2, V_3 satisfy the relations (2) and let $\bar{V}_1$, $\bar{V}_2$, $\bar{V}_3$ be such that

$$\left.\begin{aligned} & V_1 \leq \bar{V}_1 \leq V_1 + d\,(1 + \delta) \\ & \bar{V}_2 \geq V_2 - d\,(1 + \delta) \\ & \bar{V}_3 = \frac{V_o \bar{V}_2 - \bar{V}_1^2}{\bar{V}_o + \bar{V}_2 - 2\bar{V}_1} \end{aligned}\right\}$$

<u>then</u> $\bar{V}_3 > 2r - V_o$

Proof. By assumption 2: $V_3 > 2r - V_o$. By assumption (1): $V_2 < V_o$.

Let
$$x_1 = V_1 + \alpha$$
$$x_2 = V_2 - \alpha$$
$$x_3(\alpha) = \frac{V_o x_2 - x_1^2}{V_o + x_2 - 2x_1} \quad .$$

We have: $x_3(0) = V_3 > 2r - V_o$

$$x_3\left(\frac{V_o + V_2 - 2V_1}{3}\right) = -\infty$$

Since $x_3(\alpha)$ is continuous in the interval $\left[0, \frac{V_o + V_2 - 2V_1}{3}\right)$, there exists

$$0 < \beta < \frac{V_o + V_2 - 2V_1}{3} \quad \text{for which } x_3(\beta) = 2r - V_o .$$

It is easy to check that $V_1 + \beta < V_o$.

By lemma 2a, for every $V_1 \leq \overline{V}_1 < \beta$ and $\overline{V}_2 > V_2 - \beta$, one has

$$V_3 > x_3(\beta) = 2r - V_o \quad .$$

For proving the lemma 4, we have to show that $d(1 + \delta) < \beta$.

β satisfy the equation: $\dfrac{V_o(V_2 - \beta) - (V_1 + \beta)^2}{V_o + V_2 - \beta - 2(V_1 + \beta)} = 2r - V_o$.

By the well-known translation invariance of the δ^2 process, we have:

$$\frac{(V_o-r)(V_2-r-\beta) - (V_1-r+\beta)^2}{(V_o-r) + (V_2-r-\beta) - 2(V_1-r+\beta)} + (V_o - r) = 0. \qquad (4)$$

We set: $(V_1 - r) = b'(V_o - r) \qquad -b \leq b' \leq b$

$(V_3 - r) = c'(V_o - r) \qquad -c \leq c' \leq c$

Solving (4), we find: $\beta = (V_o-r) \left\{ -(2+b') + \sqrt{\dfrac{5 + 2b'^2 + 2b' - 3c' - 6b'c'}{1 - c'}} \right\}$;

the derivative of the quantity under $\sqrt{}$ with respect to c' is $\dfrac{2(1-b')^2}{(1-c')^2} > 0$;

if we replace c' by -c and $(V_o - r)$ by 1, we get the inequality:

$$\beta \geq \left\{ -(2+b') + \sqrt{\frac{5 + 2b'^2 + 2b' + 3c + 6b'c}{1 + c}} \right\};$$

We set $A = (2+b')^2$ and $B = \sqrt{}$ so that $\beta \geq \sqrt{\beta} - \sqrt{A}$.

It is easy to show that $16 > B > A$; by the mean value theorem:

$$\sqrt{B} - \sqrt{A} > \frac{1}{2\sqrt{B}}(B-A) > \frac{(1-b')^2(1-c)}{4(1+c)} \geq \frac{(1-b)^2(1-c)}{4(1+c)} = d(1+\delta) ,$$

i.e. $d(1+\delta) < \beta$ as desired.

<u>Lemma 5</u>. If $W_o > u_1 > w_1 > r$, there exists z such that

$G(z) = W_1$ and $W_o > z > W_1$.

<u>Proof</u>. $G(x)$ is a continuous function with $G(W_o) = u_1$ and $G(r) = r$;

for $u_1 > W_1 > r$, there exists $W_o > z > r$ for which $G(z) = W_1$. By assumption 1, $z > W_1$.

<u>Lemma 6</u>. For any z satisfying the conditions

$W_o \geq r + 1 > z > r$ and $G(z) = W_1$,

we have $W_2 \geq W_1$.

<u>Proof</u>. Let $$\bar{u}_3 = \frac{zu_2 - W_1^2}{z + u_2 - 2W_1} . \tag{5}$$

By assumption 2: $\bar{u}_3 > 2r - z > r - 1$;

by lemma 2d, if we replace in (5) $\bar{u}_3$ by $r - 1 < \bar{u}_3$ and z by $r + 1 > z$, letting W_1 unchanged, we have to replace u_2 by γ with $u_2 > \gamma$:

$$\frac{(r + 1)\ \gamma - W_1^2}{r + 1 + \gamma - 2W_1} = r - 1,$$

$$W_1 - \gamma = \frac{1 - (W_1 - r)^2}{2} \leq \frac{1}{2} ;$$

consequently: $u_2 > \gamma \geq W_1 - \frac{1}{2}$;

$$W_2 = W_o + [u_2 - W_o + \xi_1]^{\swarrow} \geq W_o + [W_1 - W_o - \tfrac{3}{4}]^{\swarrow} ;$$

by assumption 1: $W_1 < Z < W_o$;

therefore $W_1 - W_o < 0$ and $[W_1 - W_o - \frac{3}{4}]^{\swarrow} = W_1 - W_o$;

$W_2 \geq W_1$, q.e.d.

<u>Lemma 7</u>. If $W_o \geq r + 1 + \frac{a}{1 - b}$, then $W_2 \leq W_o$; if the sign "=" holds, $W_o - W_1 \geq 2$.

<u>Proof</u>. By assumption 1:

$$|u_1 - r| \leq b(W_o - r),$$

$$|W_1 - r| < b(W_o - r) + a + 1,$$

$$|u_2 - r| < b^2(W_o - r) + ab + b,$$

$$u_2 + \xi_1 < r + b^2(W_o - r) + ab + b + a,$$

$$u_2 + \xi_1 - W_o < (b^2 - 1)(W_o - r) + ab + b + a \leq (b^2 - 1)(1 + \frac{a}{1 - b})$$
$$+ ab + b + a = b^2 - 1 + b < 1,$$

consequently: $[u_2 + \xi_1 - W_o]^{\swarrow} \leq 0,$

$W_2 = W_o + [u_2 + \xi_1 - W_o]^{\swarrow} \leq 0,$ q.e.d.

For proving the second part of lemma 7, suppose that $W_1 = W_o$, but $W_o - W_1 < 2$. By lemma 3, the only possibility is $W_1 = W_o - 1$ for which $W_1 - r \geq \frac{a}{1 - b}$.

By assumption 1: $|u_2 - r| \leq b(W_1 - r),$

$$u_2 \leq r + b(W_1 - r),$$

$$u_2 + \xi_1 - W_1 \leq (b - 1)(W_1 - r) + a \leq (b - 1) \frac{a}{1 - b} + a = 0,$$

$$u_2 + \xi_1 \leq W_1$$

$W_2 = [u_2 + \xi_1]_R \leq W_1 < W_o$, which is a contradiction.

Lemma 8. If $W_o \geq r + 1 + \frac{a}{1 - b}$ and $W_1 \leq 2r - W_o$, then

$W_2 > W_1 > 2r - W_o - 1$

Proof. By assumption 1:

$$u_1 \geq r - b(W_o - r),$$

$$W_1 = [u_1 + \xi_o]_R > r - b(W_o - r) - a - 1 = 2r - W_o + (1 - b)(W_o - r) - a - 1;$$

$$W_1 > 2r - W_o + (1 - b)(1 + \frac{a}{1 - b}) - a - 1 = 2r - W_o - b > 2r - W_o - 1.$$

For the second part of the lemma, we use again the assumption 1:

$$u_2 \geq r - b(r - W_1),$$

$$u_2 \geq W_1 + (1 - b)(r - W_1) \geq (1 - b)(1 + \frac{a}{1 - b}) ;$$

$$u_2 + \xi_1 \geq u_2 - a \geq W_1 + 1 - b > W_1,$$

$$u_2 + \xi_1 - W_o > W_1 - W_o;$$

by lemma 3: $W_1 - W_o < 0$;

$$[u_2 + \xi_1 - W_o]^{\swarrow} > W_1 - W_o,$$

$$W_2 = W_o + [u_2 + \xi_1 - W_o]^{\swarrow} > W_1, \quad \text{q.e.d.}$$

Lemma 9. If $W_o \geq r \geq W_1$, then $W_2 \geq W_1$.

Proof. By assumption 1, $u_2 \geq W_1$;

$$u_2 + \xi_1 \geq W_1 - a \geq W_1 - \frac{1}{4} ;$$

$$W_2 = W_o + [u_2 + \xi_1 - W_o]^{\swarrow} \geq W_o + [W_1 - \frac{1}{4} - W_o]^{\swarrow} = W_1, \quad \text{q.e.d.}$$

Scheme of the proof of Theorem 1

The theorem results from the two statements:

1) If $|W_o - r| \geq 1 + \frac{a}{1 - b}$, then $|W_3 - r| < |W_o - r|$ for $r - \ell + 1 \leq W_o \leq r + \ell - 1$.

2) If $|W_o - r| < 1 + \frac{a}{1 - b}$, then $|W_3 - r| < 1 + \frac{a}{1 - b}$.

We can restrict ourselves to the case $W_o \geq r$. Indeed, suppose that the statements I) and II) are right for $W_o \geq r$ and consider a particular value $W_o < r$. We set $W_o' = -W_o$, $G'(x) = -G(-x)$, $\xi_i' = -\xi_i$ and compute W_1', W_2', W_3' by the equations 3 with the ' values. Since ξ_i', $G'(x)$, $-r$ satisfy the same hypothesis as ξ_i, $G(x)$, r, we have:

I) If $|W_o' + r| > 1 + \frac{a}{1 - b}$, then $|W_3' + r| < |W_o' + r|$ and by lemma 1 $|W_3 - r| < |W_o - r|$.

II) If $|W_o' + r| < 1 + \frac{a}{1 - b}$, then $|W_3' + r| < 1 + \frac{a}{1 - b}$ and by lemma 1 $|W_3 - r| < 1 + \frac{a}{1 - b}$.

Statement I: If $W_o \geq r + 1 + \frac{a}{1 - b}$, then $|W_3 - r| < |W_o - r|$.

Proof. We distinguish three cases:

1) $W_1 \geq W_o$

2) $W_o > W_1 > W_2$

3) $W_o > W_1$ and $W_2 \geq W_1$

1) By lemma 3, this case never occurs.

2a) $W_o > W_1 > W_2$ and $W_1 \geq u_1 = G(W_o)$.

We define $V_o = W_o$, $V_1 = u_1 = G(W_o)$, $V_2 = G(u_1)$, $V_3 = \dfrac{V_o V_2 - V_1^2}{V_o + V_2 - 2V_1}$.

Since $W_1 = W_o + [u_1 + \xi_o - W_o]$ and $W_1 < W_o$:

$W_1 \leq u_1 + a \leq u_1 + d$.

By assumption 3: $|u_2 - V_2| = |G(W_1) - G(u_1)| \leq \delta\, |W_1 - u_1| \leq \delta\, d$;

Since $W_2 = W_o + [u_2 + \xi_1 - W_o]$ and $W_2 < W_o$ we have:

$W_2 \geq u_2 - a \geq u_2 - d \geq V_2 - d(1 + \delta)$.

We set $\bar{V}_i = W_i$ and apply the lemma 4, the V_i's keeping their signification:

$W_3 > 2r - W_o$.

By lemma 2b: $u_3 < W_2$;

$W_3 = [u_3]_R \leq W_2 < W_o$ so that

$|W_3 - r| < |W_o - r|$, q.e.d.

2b) $W_o > W_1 > W_2$ and $W_1 < u_1 = G(W_o)$.

By lemma 9: $W_1 > r$.

By lemma 5, there exists Z such that

$W_o > Z > W_1$ and $W_1 = G(Z)$.

By lemma 6: $Z \geq r + 1$.

Since $W_2 = W_o + [u_2 + \xi_1 - W_o]^{\swarrow} < W_o$, we have $W_2 \geq u_2 - a \geq u_2 - d$.

We set $Z = V_o$, $W_1 = V_1$, $u_2 = V_2$, $W_1 = \bar{V}_1$, $W_2 = \bar{V}_2$ and apply the lemma 4:

$$\bar{V}_3 = \frac{ZW_2 - W_1^2}{Z + W_2 - 2W_1} > 2r - Z > 2r - W_o \;;$$

Since $W_o > Z$, by lemma 2a:

$$W_3 = \frac{W_o W_2 - W_1^2}{W_o + W_2 - 2W_1} > \bar{V}_3 > 2r - W_o \;.$$

By lemma 2b: $u_3 < W_2$;

$W_3 = [u_3]_R \leq W_2 < W_o$ so that

$|W_3 - r| < |W_o - r|$, q.e.d.

3) By lemma 7, $W_2 \leq W_o$, we distinguish:

3a) $W_o > W_2 \geq W_1 > 2r - W_o$;

3b) $W_o = W_2$;

3c) $W_o > W_2$ and $W_1 \leq 2r - W_o$.

3a) By lemma 2c: $W_2 \geq u_3 \geq W_1$; since W_1, W_2 are integers:

$W_2 \geq W_3 \geq W_1$, i.e., $|W_3 - r| < |W_o - r|$ q.e.d.

3b) By lemma 2c: $u_3 = \frac{1}{2}(W_o + W_1)$;

by lemma 7: $u_3 \leq \frac{1}{2}(2W_o - 2) = W_o - 1$;

$W_3 = [u_3]_R \leq W_o - 1 < W_o$.

By lemma 8: $W_1 > 2r - W_o - 1$;

$u_3 > r - \frac{1}{2}$;

$W_3 = W_o + [u_3 - W_o]^{\swarrow} > W_o + [r - \frac{1}{2} - W_o]^{\swarrow} \geq r - \frac{1}{2} > 2r - W_o$;

therefore: $W_o > W_3 > 2r - W_o$, i.e., $|W_3 - r| < |W_o - r|$ q.e.d.

3c) By lemma 8: $W_0 > W_2 > W_1 > 2r - W_0 - 1$;

by lemma 2c: $W_0 > W_2 > u_3 > W_1 > 2r - W_0 - 1$;

since $W_3 = W_0 + [u_3 - W_0]^{\swarrow}$:

$W_2 \geq W_3 > 2r - W_0$, i.e., $|W_3 - r| < |W_0 - r|$, q.e.d.

<u>Statement II</u>: If $r \leq W_0 < 1 + \frac{a}{1 - b}$, then $|W_3 - r| < 1 + \frac{a}{1 - b}$.

<u>Proof</u>. We distinguish two cases and describe for each of them all the possibilities without any computation:

1) $W_0 = s$;

2) $W_0 = t$ and $s - r < a$.

1) <u>$W_0 = s$</u>

1a) $W_1 = t \begin{cases} W_2 = s & : \quad W_3 = s \\ W_2 = q & : \quad W_3 = s \\ W_2 = p & : \quad W_3 = s \end{cases}$

2a) $W_1 = s$: $W_3 = s$ (u_3 does not depend on the value of W_2).

3a) $W_1 = q: \begin{cases} W_2 = t & : \quad W_3 = s \\ W_2 = s & : \quad W_3 = s \\ W_2 = q & : \quad W_3 = q \end{cases}$

4a) $W_1 = p \begin{cases} W_2 = t & : \quad W_3 = s \\ W_2 = s & : \quad W_3 = q \\ W_2 = q & : \quad W_3 = q \end{cases}$

2) <u>$W_0 = t$ and $s - r < a$</u>

2a) $W_1 = s \begin{cases} W_2 = t & : \quad W_3 = t \\ W_2 = s & : \quad W_3 = s \end{cases}$

2b) $W_1 = q \begin{cases} W_2 = t & : \quad W_3 = s \\ W_2 = s & : \quad W_3 = s \end{cases}$

$$2c)\quad W_1 = p \begin{cases} W_2 = t & : \quad W_3 = s \\ W_2 = s & : \quad W_3 = s \\ W_2 = q & : \quad W_3 = q \end{cases}$$

Proof of the Theorem 2

We shall use the same notations as in the proof of the theorem 1.

Lemmas

The following lemmas are valid only with the assumptions of the theorem 2.

Lemma 10. If $W_o \geq t$, then $W_1 \leq s$; the sign "=" holds only if $s - r \leq \frac{1}{3}$.

Proof. By assumption 1: $u_1 \leq r$;

$u_1 + \xi_o \leq r + \frac{1}{3} < t$;

$W_1 = W_o + [u_1 + \xi_o - W_o]^{\nearrow} < t$;

W_o can equal s only if $r + \frac{1}{3} \geq s$, i.e., $s - r \leq \frac{1}{3}$, q.e.d.

Lemma 11. If $W_o \geq r + 1$, then $W_2 \geq W_1$.

Proof. a) Suppose $W_1 \leq q$; then:

$u_2 \geq r$,

$u_2 + \xi_1 > p$,

$W_2 = W_o + [u_2 + \xi_1 - W_o]^{\swarrow} > p \geq q \geq W_1$, q.e.d.

b) Suppose $W_1 \geq s$; since $W_o \geq t$, by lemma 10, $W_1 = s$ and $s - r \leq \frac{1}{3}$;

by assumption 1: $u_2 > r - \frac{1}{3} \geq q + \frac{1}{3}$;

by assumption 2: $u_2 + \xi_1 > q$;

$W_2 = W_o + [u_2 + \xi_1 - W_o]^{\swarrow} > q$ i.e.,

$W_2 \geq s = W_1$ q.e.d.

Lemma 12. If $W_0 \geq r$, then $W_2 \leq W_0 + 1$.

Proof. By assumption 1: $|u_1 - r| < (W_0 - r)$;

$|W_1 - r| = |[u_1 + \xi_0]_R - r| < |u_1 - r| + 4/3 < (W_0 - r) + 4/3$;

$|u_2 - r| < |W_1 - r| < (W_0 - r) + 4/3$;

$u_2 < W_0 + 4/3$;

$u_2 + \xi_1 < W_0 + 5/3$;

$W_2 = W_0 + [u_2 + \xi_1 - W_0]^{\swarrow} \leq W_0 + [5/3]^{\swarrow} = W_0 + 1$, q.e.d.

Lemma 13. If $W_2 = W_0 + 1$ and $W_1 \leq W_0 - 3$, then $W_3 < W_0$.

Proof. If we keep W_0 constant, u_3 is an increasing function of W_1 by lemma 2a; since $W_3 = W_0 + [u_3 - W_0]^{\swarrow}$, W_3 will have the same property and it suffices to prove the lemma for the case $W_1 = W_0 - 3$; one finds:

$u_3 = W_0 - 9/7 < W_0 - 1$

$W_3 = [u_3]_R \leq W_0 - 1 < W_0$, q.e.d.

Lemma 14. If $W_2 = W_0 \geq r + 1 - a$, then $W_1 < r$ and $W_3 < W_0$.

Proof. Suppose $W_1 \geq r$;

by assumption 1: $u_2 \leq r$,

$u_2 + \xi_1 \leq r + a \leq W_0 - 1$,

$W_2 = [u_2 + \xi_1]_R \leq W_0 - 1$, which contradicts our hypothesis.

We prove the second part of the lemma; since $W_1 < r$, we have:

$W_0 - W_1 > 1$, i.e., $W_0 - W_1 \geq 2$;

by lemma 2c: $u_3 = \frac{1}{2}(W_0 + W_1) = W_0 + \frac{1}{2}(W_1 - W_0) \leq W_0 - 1$;

$W_3 = [u_3]_R \leq W_0 - 1 < W_0$, q.e.d.

Lemma 15. If $W_2 > W_0 \geq r + 1 + a$, then $W_1 < r - 1$ and $W_3 < W_0$.

Proof. Suppose $W_1 \geq r - 1$;

by assumption 1: $u_2 < r + 1$,

$u_2 + \xi_1 < r + 1 + a \leq W_o$,

$W_2 = [u_2 + \xi_1]_R \leq W_o$, which contradicts our hypothesis.

We prove the second part of the lemma; since $W_1 < r - 1$, we have:

$W_o - W_1 > 2$, i.e., $W_o - W_1 \geq 3$;

by lemma 12: $W_2 = W_o + 1$;

by lemma 13: $W_3 < W_o$, q.e.d.

Lemma 16. If $W_o \geq r + 4/3$, then $W_3 > 2r - W_o$.

Proof. a) Suppose $W_1 \geq r$;

by lemma 11: $W_2 \geq W_1 \geq r$;

by lemma 2b: $u_3 \geq W_1$,

$W_3 = [u_3]_R \geq W_1 \geq r > 2r - W_o$, q.e.d.

b) Suppose $W_1 < r$; we have the inequalities:

$|u_1 - r| < (W_o - r)$,

$u_1 > 2r - W_o$,

$u_1 + \xi_o > 2r - W_o - \frac{1}{3}$,

$W_1 = [u_1 + \xi_o]_R > 2r - W_o - 4/3 \equiv \bar{W}_1$.

$u_2 \geq r$,

$u_2 + \xi_1 \geq r - \frac{1}{3}$

$W_2 = W_o + [u_2 + \xi_1 - W_o]^{\swarrow} \geq W_o + [r - \frac{1}{3} - W_o]^{\swarrow} \geq r - \frac{1}{3} \equiv \bar{W}_2$.

By lemma 11, $W_2 \geq W_1$; by lemma 2a, for fixed W_o, u_3 decreases when W_2 and W_1 decrease; consequently:

$$u_3 = \frac{W_o W_2 - W_1^2}{W_o + W_2 - 2W_1} > \frac{W_o \bar{W}_2 - \bar{W}_1^2}{W_o + \bar{W}_2 - 2\bar{W}_1} \equiv B .$$

The fact that $B > 2r - W_o$ for $W_o \geq r + \frac{4}{3}$ results from the three statements:

1) B is a continuous function of W_o for $r \leq W_o < \infty$;

2) for $W_o \rightarrow \infty$, $W_3 \sim \frac{-W_o}{3}$ and therefore $B > 2r - W_o$ when W_o is large enough.

3) $W_o = r + \frac{1 + \sqrt{33}}{6} < r + \frac{4}{3}$, is the only value in $[r, \infty]$ for which $B = 2r - W_o$.

We have therefore established that $u_3 > 2r - W_o$ for $W_o \geq r + \frac{4}{3}$. Now:

$W_3 = W_o + [u_3 - W_o] > 2r - W_o$, q.e.d.

Lemma 17. If $W_o \geq r + 1 + a$, then $W_2 > 2r - W_o$.

Proof. By lemma 16, we have only to establish the lemma for $W_o < r + \frac{4}{3}$, i.e., $W_o = t$, $s - r < \frac{1}{3}$.

a) Suppose $W_1 \geq r$; by the same argument used in lemma 16, we conclude that $W_3 > 2r - W_o$.

b) Suppose $W_1 < r$;

$u_1 > 2r - t > t - \frac{8}{3}$,

$u_1 + \xi_1 > t - \frac{9}{3} = p$,

$W_1 = [u_1 + \xi_o]_R \geq p$.

$u_2 \geq r$,

$u_2 + \xi_1 \geq r - \frac{1}{3} > q + \frac{1}{3}$;

$W_2 = W_o + [u_2 + \xi_1 - W_o] > q + \frac{1}{3}$, i.e., $W_2 \geq s$;

by lemma 11: $W_2 \geq W_1$; by lemma 2a:

$$u_3 = \frac{W_o W_2 - W_1^2}{W_o + W_2 - 2W_1} \geq \frac{ts - p^2}{t + s - 2p} = q + \frac{1}{5} ;$$

therefore: $W_3 \geq s \geq r > 2r - W_o$, q.e.d.

Scheme of the proof of the theorem 2

The theorem results from the two statements:

1) If $|W_o - r| \geq 1 + a$, then $|W_3 - r| < |W_o - r|$ for $r - \ell + \frac{4}{3} \leq W_o \leq r + \ell - \frac{4}{3}$.

2) If $|W_o - r| < 1 + a$, then $|W_3 - r| < 1 + a$.

Using the same argument as in the proof of the theorem 1, we can restrict ourselves to the case $W_o \geq r$.

Statement I: If $W_o \geq r + a$, then $|W_3 - r| < W_o - r$, i.e.,

1) $W_3 < W_o$

2) $W_3 > 2r - W_o$.

1) We prove that $W_3 < W_o$; we distinguish three cases:

a) $W_2 < W_o$; by lemma 11, $W_2 \geq W_1$; by lemma 2c, $u_3 \leq W_2$; since $W_3 = [u_3]_R$, $W_3 \leq W_2 < W_o$.

b) $W_2 = W_o$; by lemma 14, $W_3 < W_o$.

c) $W_2 > W_o$; by lemma 15, $W_3 < W_o$.

2) By lemma 17, $W_2 > 2r - W_o$.

Statement II. If $r \leq W_o < r + 1 + a$, then $|W_3 - r| < 1 + a$.

We distinguish two cases; for each of them, we describe all the possibilities without any computations:

1) $W_o = s$

2) $W_o = t$ and $s - r < a$.

1) $W_o = s$

1a) $W_1 = t$ (in this case $s - r < a \leq \frac{1}{3}$) $\begin{cases} W_2 = s & : \quad W_3 = s \\ W_2 = q & : \quad W_3 = s \end{cases}$

1b) $W_1 = s$: $W_3 = s$ (u_3 does not depend on W_2)

1c) $W_1 = q \begin{cases} W_2 = s & : \quad W_3 = s \\ W_2 = q & : \quad W_3 = q \end{cases}$

1d) $W_1 = p \begin{cases} W_2 = s & : \quad W_3 = q \\ W_2 = q & : \quad W_3 = q \end{cases}$

2) $\underline{W_o = t \text{ and } s - r < a}$

2a) $W_1 = s \begin{cases} W_2 = t & : \quad W_3 = t \\ W_2 = s & : \quad W_3 = s \end{cases}$

2b) $W_1 = q \begin{cases} W_2 = t & : \quad W_3 = s \\ W_2 = s & : \quad W_3 = s \end{cases}$

2c) $W_1 = p \begin{cases} W_2 = t + 1 & : \quad W_3 = s \\ W_2 = t & : \quad W_3 = s \\ W_2 = s & : \quad W_3 = s \end{cases}$